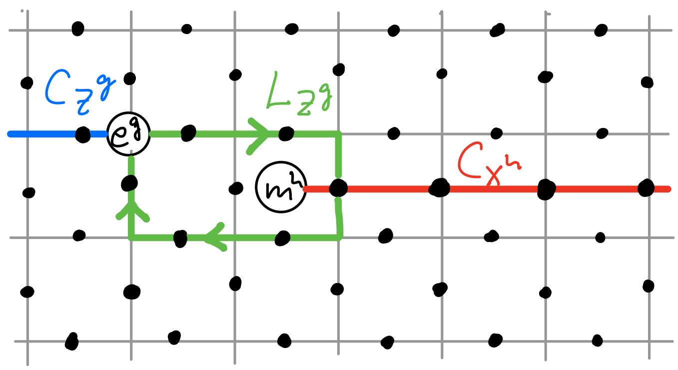
Second look at braiding:



We cannot directly exchange the e^g and m^h anyons, but we can compute the "monodromy" of taking one around the other!

$\rightarrow$ Consider the initial state

$$|\Psi_{ini}\rangle = C_{X^h} C_{Z^g} |\xi\rangle$$

strings of Pauli operators

vacuum (groundstate)

$\rightarrow$ move e^g around m^h along green path by successive applications of Z^g-rotations

$\rightarrow$ loop operator L_{Z^g}

L_{Z^g} meets C_{X^h} at a single point

$\rightarrow L_{Z^g} C_{X^h} = \omega^{gh} C_{X^h} L_{Z^g}$

We then obtain as final state:

$$|\psi_{fin}\rangle = L_{Z^g}|\psi_{ini}\rangle = L_{Z^g}\left((C_{X^h} C_{Z^g})|\xi\rangle\right)$$

$$= \omega^{gh} C_{X^h} C_{Z^g} \underbrace{L_{Z^g}|\xi\rangle}_{=|\xi\rangle}$$

as L_{Z^g} is stabilizer

$$= \omega^{gh}|\psi_{ini}\rangle$$

$\rightarrow$ we obtain as monodromy:

$$M = \omega^{gh}, \quad \omega = e^{2\pi i/d}$$

$$\shortparallel$$

$(R)^2$ where R is the exchange operator of two anyons

$$\Rightarrow R = e^{\pi i \frac{gh}{d}}$$

Result is independent of shape of L_{Z^g} loop as long as it circulates m^h-anyon exactly once!

Example II: The non-Abelian $D(S_3)$ model

We take G to be simplest non-Abelian finite group: $G = S_3$

$$S_3 = \{e, c, c^2, t, tc, tc^2\}$$

idetity cyclic perm. exchange of (1,2)

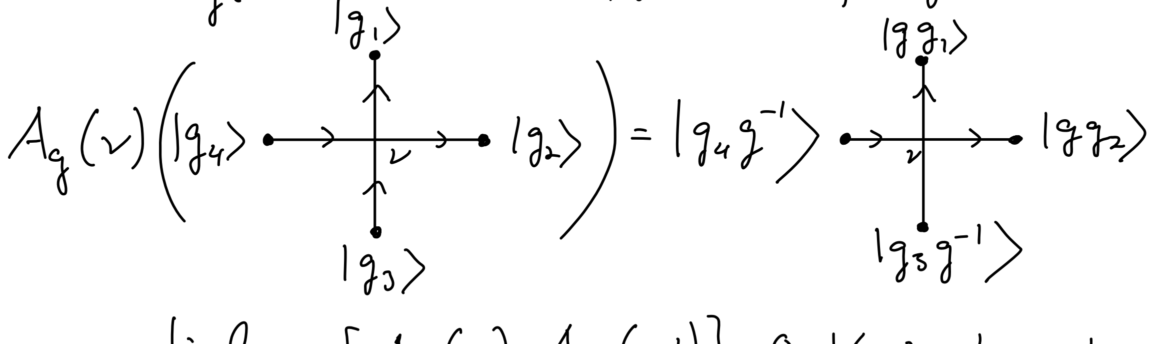
we have: $t^2 = c^3 = e, \quad tc = c^2 t$

$\longrightarrow |S_3| = 6$

Pick oriented two-dimensional square lattice $\longrightarrow$ assign 6-level spin spanned by states $|g\rangle$ to each edge

Define operators acting on vertex v by:

$$A_g(v) = L^g_{+,1} L^g_{+,2} L^g_{-,3} L^g_{-,4} \quad \text{for } g \in S_3$$



$\longrightarrow$ satisfy $[A_g(v), A_{g'}(v')] = 0 \quad \forall\, v, v', g, g'$

Creation operators:

$$W_\Lambda(s) = |e\rangle\langle e| + |c\rangle\langle c| + |c^2\rangle\langle c^2| - |t\rangle\langle t| - |tc\rangle\langle tc| - |tc^2\rangle\langle tc^2| \qquad (*)$$

$$W_\Phi(s) = 2|e\rangle\langle e| - |c\rangle\langle c| - |c^2\rangle\langle c^2|$$

$\rightarrow$ can be checked by applying P_Λ and P_Φ

<u>Fusion rules</u>:

$$\Lambda \times \Lambda = 1, \quad \Lambda \times \Phi = \Phi, \quad \Phi \times \Phi = 1 + \Lambda + \Phi$$

$\longrightarrow$ Φ is non-Abelian anyon!

(there are more anyons in $D(S_3)$, but we focus here on closed sub-algebra $1, \Lambda, \Phi$)

Verification:

- $W_\Lambda(s) W_\Phi(s) = W_\Phi(s)$
- $W_\Phi(s) W_\Phi(s) = 4|e\rangle\langle e| + |c\rangle\langle c| + |c^2\rangle\langle c^2|$
  $= W_1(s) + W_\Lambda(s) + W_\Phi(s)$

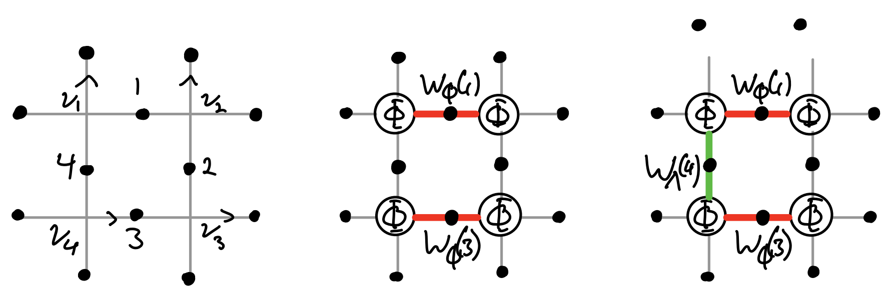
Non-Abelian information encoding and manipulation

The $D(S_3)$ model offers simple subset of particles, $1, \Lambda, \Phi$, satisfying

$$\Lambda \times \Lambda = 1, \quad \Lambda \times \Phi = \Phi, \quad \Phi \times \Phi = 1 + \Lambda + \Phi$$

$\longrightarrow$ employ last fusion rule to encode qubit states in fusion outcomes 1 and Λ

To encod a qubit, consider 4 neighbouring vertices:



vacuum $|\xi\rangle$ $\qquad$ $W_\Phi(1) W_\Phi(2) |\xi\rangle$ $\qquad$ $W_\Lambda(4) W_\Phi(1) W_\Phi(3) |\xi\rangle$

carry vacuum fusion channel $\qquad$ carry Λ fusion ch.

$\longrightarrow$ define logical qubits:

$$|0_L\rangle = W_\Phi(1) W_\Phi(3) |\xi\rangle$$

$$|1_L\rangle = W_\Lambda(4) W_\Phi(1) W_\Phi(3) |\xi\rangle$$

Properties:

1) Both logical states are composed of 4 Φ anyon states, but with different pairwise fusion channels!

2) possible to move encoding Φ anyons apart without destroying fusion outcome

→ encoded information is topologically protected from local perturbations/errors

To position anyons further apart, separated by a chain of spins C, use:

$$W_\Lambda(C) = \prod_{s \in C} W_\Lambda(s)$$

$$W_\Phi(C) = \sum_{k=0,1,2} \sum_{g_n \times \dots \times g_1 = c^k} (\omega^k + \omega^{-k}) |g_1, \dots, g_n\rangle\langle g_1, \dots, g_n|$$

where $g_1, \dots, g_n$ are states of spins within the chain C, $c \in S_3$, $\omega = e^{2\pi i/3}$.

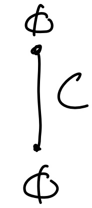
→ for $n=1$ one recovers definition (*)

→ by employing $4n$ anyons of type Φ, we can encode n qubits!

Logical operators:

- a logical X operation corresponds to creating two Λ charges and fusing both with a ϕ from each pair:

$$X = W_\Lambda(C) \, ,$$



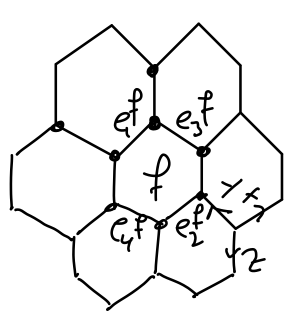
- logical Z operation corresponds to vertex operators acting on both ϕ charges of either pair:

$$Z = A_t(v_1)\, A_t(v_2)$$

Kitaev's Honeycomb model

Consider the following two-body nearest neighbor model:

$$H = -J_x \sum_{(i,j)\in E_x} X_i X_j - J_y \sum_{(i,j)\in E_y} Y_i Y_j - J_z \sum_{(i,j)\in E_z} Z_i Z_j$$

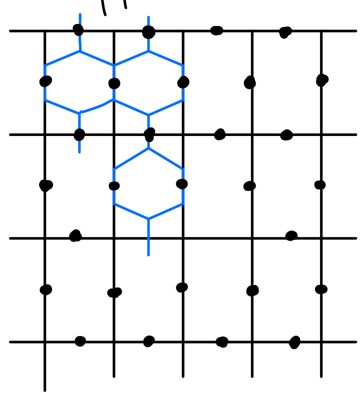


$$\xrightarrow{J_x, J_y \ll J_z} \quad H_{eff} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_f \widetilde{X}_{e_1 f} \widetilde{X}_{e_2 f} \widetilde{Y}_{e_3 f} \widetilde{Y}_{e_4 f} - J_z \sum_{e\in E_z} \widetilde{Z}_e$$

where $\bar{A}_e = A_i A_j$ with $e = (i,j)$, $A = X, Y, Z$

Let us define qubit $\{|\bar{0}\rangle = |00\rangle, |\bar{1}\rangle = |11\rangle\}$ for each vertical edge, stabilized in $J_z \gg 0$ limit, and Pauli operators $\bar{X} = XX$ and $\bar{Z} = Z \otimes I$

$\longrightarrow$ effective lattice



Hamiltonian can be reformulated as

$$\bar{H}_{eff} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_f \bar{X}_{e_l^f} \bar{X}_{e_r^f} \bar{Z}_{e_t^f} \bar{Z}_{e_b^f}$$

$\longrightarrow$ Hadamard transformation on all horizontal edges gives Kitaev's toric code Hamiltonian!